Schlesinger system, Einstein equations and hyperelliptic curves

Dedicated to the memory of Moshé Flato

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Abstract. We review recent developments in the method of algebro-geometric integration of integrable systems related to deformations of algebraic curves. In particular, we discuss the theta-functional solutions of Schlesinger system, Ernst equation, and self-dual $SU(2)$-invariant Einstein equations.

1 Introduction

Deep interaction between algebraic geometry of the compact Riemann surfaces and the theory of integrable systems represent nowadays a well established paradigm of modern mathematical physics. The analysis on the compact Riemann surfaces was essentially completed in 19th century in the famous classical works by Riemann, Abel, Jacobi, Weierstrass etc. The penetration of the tools provided by their methods in the theory of integrable systems started at the same time. Lagrange first solved the equations of motion of the Euler top by means of the use of the Jacobi elliptic functions. Then Carl Neumann solved in terms of theta-functions the equations of geodesic motion on ellipsoid [3]. Slightly later Dobriner published the explicit solution of sine-Gordon equation expressed in terms of two-dimensional theta-functions [2]. The essential breakthrough was achieved in 1889 by Kowalevski [1] who found the new integrable case of the motion of rigid body solvable in terms of genus 2 algebraic functions.

The same time it started the development of the spectral theory of the Sturm-Liouville operators with periodic coefficients. First important steps in the field were done by Floquet and Lyapunov. In 1919 french matematician Jule Drach wrote the remarkable article (absolutely forgotten for next 60 years) devoted to the construction of explicitly solvable Sturm-Liouville equations associated to hyperelliptic curves of any genus [4].

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One of the subsequent developments of the discovery of the inverse scattering transform (IST) method by Gardner, Green, Kruskal and Miura [5] was the complete understanding of the deep interplay between the classical works listed above. This connection now became a part of what is called algebrogeometric approach to the solution of nonlinear differential equations. This field of activity was initiated in 1974-1976 by the works of Novikov, Lax, M.Kac, Its, Matveev, Dubrovin, McKean, van Moerbeke and Krichever.

Speaking about the aspects of the theory connected to the explicit solutions of these equations, one should mention the formulas for solutions of the Korteveg-deVries equation with periodic initial data obtained in [6]. Soon after the same kind of formulas were derived for numerous other equations integrable via IST method, including Non-linear Schrödinger [7], Sine-Gordon [8], Toda lattice [9], Kadomtzev-Petviashvili [10] and so on. The generic feature of the algebro-geometric solutions of the systems listed above is that they are parametrised by the fixed algebraic curves and the associated dynamics is linear on their Jacobians. These results found many beautiful and unexpected applications in various branches of modern mathematics and physics, including differential geometry of surfaces [11] and algebraic geometry (Novikov conjecture [12]).

It is also relevant to notice that simultaneously with the soliton theory, the algebro-geometric methods were actively applied in the framework of twistor theory. Probably, the main achievements in this direction were classifications of the instanton and self-dual monopole configurations. More recently twistor methods were applied by Hitchin to classify self-dual $SU(2)$ invariant Einstein manifolds [13]. It turned out that the description of most non-trivial subclass of these manifolds gives rise to some (solvable in terms of elliptic functions) partial case of four-pole Schlesinger system (or, equivalently, Painlevé 6 equation).

Technically and conceptually new development was the application of algebrogeometric ideology to the Einstein equations of general relativity in presence of two commuting Killing vectors (Ernst equation). The embedding of this equation in the framework of the IST approach was initiated by Belinskii-Zakharov [14] and Maison [15] [3]. The characteristic feature of the associated zero-curvature representation is the non-trivial dependence of the associated connection on the spectral parameter. Namely, the connection lives on the genus zero algebraic curve depending on space-time variables. This peculiarity entails the drastic change of the construction and qualitative properties of algebro-geometric solutions which were first obtained in 1988 [17, 18]. Dynamics in these solutions is generated by hyperelliptic curves with two coordinate-dependent branch points. The algebro-geometric solutions of the Ernst equation do not possess any periodicity properties which were inevitable for all KdV-like cases studied before. Moreover, we can explicitly incorporate in the construction an arbitrary functional parameter which never appears in the traditional KdV-like 1+1 integrable systems. Wide subclass of the obtained solutions turns out to be asymptotically flat [19]. As a simple degenerate case the algebro-geometric solutions contain the whole class of multisoliton solutions found by Belinskii and Zakharov [14].

Simplest elliptic solutions were studied in [19]. Despite many interesting properties, they contain ring-like naked singularities making it difficult to exploit them in a real physical context. Realistic physical application of the particular family of the class of algebro-geometric solutions came out from the series of papers of Meinel, Neugebauer and

[3] Coming back to the works of classics of differential geometry of 19th century it is curious to mention that essentially the same Lax pair appeared in the work of Bianchi [16] in the study of so-called Bianchi congruences.

their collaborators starting from 1993 [20]. These works were devoted to the investigation of the boundary-value problem corresponding to the infinitely thin rigid relativistically-rotating dust disc. The embedding of the dust disc solution in the formulas of [17] as a special solution of genus 2 was described in [21].

The recent work [24] provided the unifying framework for results of [17, 13]. In this paper it was shown that an arbitrary 2×2 inverse monodromy problem with off-diagonal monodromy matrices admits an explicit solution in terms of multi-dimensional theta-functions.[4] In the subsequent paper [29] results of [24] were applied to self-dual $SU(2)$ invariant Einstein manifolds, which allowed to considerably simplify the original formulas of Hitchin [13].

Existence of close link between the Schlesinger system and Ernst equation, revealed in [26], allowed to apply results of [24] also to Ernst equation [33] which gave a simplification of original description of [17] and allowed to integrate equations for all metric coefficients associated to algebro-geometric Ernst potentials.

Our purpose here is to describe results of [24, 29, 33] in unified framework.

2 Solutions of Schlesinger system in terms of theta-functions

Let us briefly remind the origin of the Schlesinger system. Consider the following linear differential equation for function $\Psi(\lambda)\in SL(2,\mathbb{C})$:

$$\frac{d\Psi}{d\gamma}=\sum_{j=1}^{N}\frac{A_j}{\gamma-\gamma_j}\Psi\ , \tag{2.1}$$

where matrices $A_j\in sl(2,\mathbb{C})$ are independent of γ. Let us impose the initial condition $\Psi(\gamma=\infty)=I$. Function $\Psi(\gamma)$ defined in this way lives on the universal covering X of $\mathbb{C}P^1\setminus\{\gamma_1,\dots,\gamma_N\}$. The asymptotical expansion of $\Psi(\gamma)$ near singularities γ_j is given by

$$\Psi(\gamma)=(Q_j+O(\gamma-\gamma_j))(\gamma-\gamma_j)^{T_j}C_j\ , \tag{2.2}$$

where T_j is traceless diagonal matrix; $Q_j,\ C_j\ \in SL(2,\mathbb{C})$. Matrices

$$M_j=C_j^{-1}e^{2\pi i T_j}C_j,\qquad j=1,\dots,N \tag{2.3}$$

are called the monodromy matrices of system (2.1).

The assumption of independence of all matrices M_j of the positions of the poles γ_j: $\partial M_j/\partial\gamma_k=0$ is called the isomonodromy condition; it implies the following dependence of $\Psi(\gamma)$ on γ_j:

$$\frac{\partial\Psi}{\partial\gamma_j}=-\frac{A_j}{\gamma-\gamma_j}\Psi\ . \tag{2.4}$$

The compatibility condition of (2.1) and (2.4) is equivalent to the Schlesinger system [23] for the residues A_j:

$$\frac{\partial A_j}{\partial\gamma_i}=\frac{[A_i,A_j]}{\gamma_i-\gamma_j},\quad i\neq j\ ,\qquad \frac{\partial A_i}{\partial\gamma_i}=-\sum_{j\neq i}\frac{[A_i,A_j]}{\gamma_i-\gamma_j}. \tag{2.5}$$

[4] Almost simultaneously the same inverse monodromy problem was solved in the paper [25] motivated by random matrix theory.

The tau-function $\tau(\{\gamma_j\})$ of the Schlesinger system is defined by equations

$$\frac{\partial}{\partial\gamma_j}\log\{\tau\} = \sum_{k\neq j}\frac{\mathrm{tr}A_jA_k}{\gamma_j-\gamma_k}\,. \tag{2.6}$$

Compatibility of equations (2.6) follows from the Schlesinger system.

Certain class of functions Ψ, satisfying isomonodromy conditions, was constructed in the paper [24]. To describe this construction let us take $N = 2g+2$ and introduce the hyperelliptic curve $\mathcal{L}$ of genus γ by the equation

$$w^2 = \prod_{j=1}^{2g+2}(\gamma-\gamma_j) \tag{2.7}$$

with branch cuts $[\gamma_{2j+1},\gamma_{2j+2}]$. Let us choose the canonical basis of cycles (a_j,b_j), $j = 1,\dots,g$ such that the cycle a_j encircles the branch cut $[\gamma_{2j+1},\gamma_{2j+2}]$. Cycle b_j starts from one bank of branch cut $[\gamma_1,\gamma_2]$, goes to the second sheet through of branch cut $[\gamma_{2j+1},\gamma_{2j+2}]$, and comes back to another bank of the branch cut $[\gamma_1,\gamma_2]$.

The dual basis of holomorphic 1-forms on $\mathcal{L}$ are given by $\frac{\gamma^{k-1}d\gamma}{w}$, $k = 1,\dots,g$.

Let us introduce two $g\times g$ matrices of a- and b-periods of these 1-forms:

$$\mathcal{A}_{kj} = \oint_{a_j}\frac{\gamma^{k-1}d\gamma}{w}, \qquad \mathcal{B}_{kj} = \oint_{b_j}\frac{\gamma^{k-1}d\gamma}{w}. \tag{2.8}$$

The holomorphic 1-forms

$$dU_k = \frac{1}{w}\sum_{j=1}^{g}(\mathcal{A}^{-1})_{kj}\gamma^{j-1}d\gamma \tag{2.9}$$

satisfy the normalization conditions $\oint_{a_j} dU_k = \delta_{jk}$.

The matrices $\mathcal{A}$ and $\mathcal{B}$ define the symmetric $g\times g$ matrix of b-periods of the curve $\mathcal{L}$: $\mathbf{B} = \mathcal{A}^{-1}\mathcal{B}$. Now we can also define the g-dimensional theta-function $\Theta[^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z}|\mathbf{B})$, associated to curve $\mathcal{L}$, with argument $\mathbf{z}\in\mathbb{C}^g$ and arbitrary complex characteristic $[^{\mathbf{p}}_{\mathbf{q}}]$ ($\mathbf{p}\in\mathbb{C}^g$, $\mathbf{q}\in\mathbb{C}^g$). The theta-function has standard periodicity properties with respect to the shift of the argument on any integer linear combination of vectors $\mathbf{e}_j \equiv (0,\dots,1,\dots,0)^t$ (1 stands in the jth place) and $\mathbf{B}\mathbf{e}_j$.

Let us cut the curve $\mathcal{L}$ along all basic cycles to get the fundamental polygon $\hat{\mathcal{L}}$. For any meromorphic 1-form dW on $\mathcal{L}$ we can define the integral $\int_Q^P dW$, where the integration contour lies inside of $\hat{\mathcal{L}}$ (if dW is meromorphic, the value of this integral might also depend on the choice of integration contour inside of $\hat{\mathcal{L}}$). The vector of Riemann constants corresponding to our choice of the initial point of this map is given by the formula (see [27]) $K_j = \frac{j}{2}+\frac{1}{2}\sum_{k=1}^{g}\mathbf{B}_{jk}$.

The characteristic with components $\mathbf{p}\in\mathbb{C}^g/2\mathbb{C}^g$, $\mathbf{q}\in\mathbb{C}^g/2\mathbb{C}^g$ is called half-integer characteristic: the half-integer characteristics are in one-to-one correspondence with the half-periods $\mathbf{B}\mathbf{p}+\mathbf{q}$. To any half-integer characteristic we can assign parity which by definition coincides with the parity of the scalar product $4\langle\mathbf{p},\mathbf{q}\rangle$.

The odd characteristics which will be of importance for us in the sequel correspond to any given subset $S = \{\gamma_{i_1},\dots,\gamma_{i_{g-1}}\}$ of $g-1$ arbitrary non-coinciding branch points. The odd half-period associated to the subset S is given by

$$\mathbf{B}\mathbf{p}^S+\mathbf{q}^S = \sum_{j=1}^{g-1}\int_{\gamma_1}^{\gamma_{i_j}} dU - K \tag{2.10}$$

where $dU = (dU_1, \dots, dU_g)^t$. Denote by $\Omega_\gamma \subset \mathbb{C}$ the neighbourhood of the infinite point $\gamma = \infty$, such that Ω_γ does not overlap with projections of all basic cycles on γ-plane. Let the 2×2 matrix-valued function $\Phi(\gamma)$ be defined in the domain Ω_γ of the first sheet of $\mathcal{L}$ by the following formula,

$$\Phi(\gamma \in \Omega_\gamma) = \begin{pmatrix} \varphi(\gamma) & \varphi(\gamma^*) \\ \psi(\gamma) & \psi(\gamma^*) \end{pmatrix}, \tag{2.11}$$

where functions φ and ψ are defined in the fundamental polygon $\hat{\mathcal{L}}$ by the formulas:

$$\varphi(\gamma) = \Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} \left(\int_{\gamma_1}^{\gamma} dU + \int_{\gamma_1}^{\gamma_\varphi} dU \middle| \mathbf{B} \right) \Theta \begin{bmatrix} \mathbf{p}^S \\ \mathbf{q}^S \end{bmatrix} \left(\int_{\gamma_\varphi}^{\gamma} dU \middle| \mathbf{B} \right), \tag{2.12}$$

$$\psi(\gamma) = \Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} \left(\int_{\gamma_1}^{\gamma} dU + \int_{\gamma_1}^{\gamma_\psi} dU \middle| \mathbf{B} \right) \Theta \begin{bmatrix} \mathbf{p}^S \\ \mathbf{q}^S \end{bmatrix} \left(\int_{\gamma_\psi}^{\gamma} dU \middle| \mathbf{B} \right), \tag{2.13}$$

with two arbitrary (possibly $\{\gamma_j\}$-dependent) points γ_φ, $\gamma_\psi \in \mathcal{L}$ and arbitrary constant complex characteristic $\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix}$; $*$ is the involution on $\mathcal{L}$ interchanging the sheets. An odd theta characteristic $\begin{bmatrix} \mathbf{p}^S \\ \mathbf{q}^S \end{bmatrix}$ corresponds to an arbitrary subset S of $g-1$ branch points via Eq. (2.10).

Since domain Ω_γ does not overlap with projections of all basic cycles of $\mathcal{L}$ on γ-plane, domain $\mathcal{L}_\gamma^*$ does not overlap with the boundary of $\hat{\mathcal{L}}$, and functions $\varphi(\gamma^*)$ and $\psi(\gamma^*)$ in (2.11) are uniquely defined by (2.12), (2.13) for $\gamma \in \Omega_\gamma$.

Now choose some sheet of the universal covering X, define new function $\Psi(\gamma)$ in subset Ω_γ of this sheet by the formula

$$\Psi(\gamma \in \Omega_\gamma) = \sqrt{\frac{\det \Phi(\infty^1)}{\det \Phi(\gamma)}} \Phi^{-1}(\infty^1) \Phi(\gamma) \tag{2.14}$$

and extend on the rest of X by analytical continuation.

Function $\Psi(\gamma)$ (2.14) transforms as follows with respect to the tracing around basic cycles of $\mathcal{L}$ (by T_{a_j} and T_{b_j} we denote corresponding operators of analytical continuation):

$$T_{a_j}[\Psi(\gamma)] = \Psi(\gamma) e^{2\pi i p_j \sigma_3} \; ; \qquad T_{b_j}[\Psi(\gamma)] = \Psi(\gamma) e^{-2\pi i q_j \sigma_3}$$

(by σ_j , $j = 1, 2, 3$ we denote standard Pauli matrices).

The following statement proved in the paper [24] claims that function Ψ satisfies condition of isomonodromy, and, therefore, provides a class of solutions of Schlesinger system:

Theorem 2.1 *Let $\mathbf{p}, \mathbf{q} \in \mathbb{C}^g$ be an arbitrary set of $2g$ constants such that characteristic $\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix}$ is not half-integer. Then:*

1. *Function $\Psi(Q \in X)$ defined by (2.14) is independent of γ_φ and γ_ψ, and satisfies the linear system (2.1) with*

$$A_j \equiv \mathrm{res}|_{\gamma=\gamma_j} \left\{ \Psi_\gamma \Psi^{-1} \right\}, \tag{2.15}$$

which in turn solve the Schlesinger system (2.5).

2. *Monodromies (2.3) of $\Psi(\gamma)$ around points γ_j are given by*

$$M_j = \begin{pmatrix} 0 & -m_j \\ m_j^{-1} & 0 \end{pmatrix} \ , \tag{2.16}$$

where constants m_j may be simply expressed in terms of $\mathbf{p}$ and $\mathbf{q}$ (see [24]).

3. *The τ-function, corresponding to solution (2.15) of the Schlesinger system, has the following form:*

$$\tau(\{\gamma_j\}) = [\det \mathcal{A}]^{-\frac{1}{2}} \prod_{j<k} (\gamma_j - \gamma_k)^{-\frac{1}{8}} \Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} (0|\mathbf{B}) \ . \tag{2.17}$$

3 Algebro-geometric solutions of Ernst equation

Here we are going to describe the application of the construction of previous section to the Einstein equations with two commuting Killing vectors. The metric on stationary axially symmetric space-time with coordinates (ρ, z, φ, t) may be represented in the following form:

$$ds^2 = f^{-1}[e^{2k}(dz^2 + d\rho^2) + \rho^2 d\varphi^2] - f(dt + F d\varphi)^2 \ , \tag{3.1}$$

where all metric coefficients f, k, F are assumed to depend only on ρ and z. For this kind of metric the essential part of the Einstein equations is the Ernst equation

$$(\mathcal{E} + \bar{\mathcal{E}})(\mathcal{E}_{zz} + \frac{1}{\rho}\mathcal{E}_\rho + \mathcal{E}_{\rho\rho}) = 2(\mathcal{E}_z^2 + \mathcal{E}_\rho^2) \tag{3.2}$$

for complex-valued function $\mathcal{E}(z, \rho)$ - the so-called Ernst potential. The metric coefficients may be restored from $\mathcal{E}(z, \rho)$ in quadratures according to the following equations:

$$f = \Re \mathcal{E} \ , \qquad F_\xi = 2\rho \frac{(\mathcal{E} - \bar{\mathcal{E}})_\xi}{(\mathcal{E} + \bar{\mathcal{E}})^2} \ , \qquad k_\xi = 2i\rho \frac{\mathcal{E}_\xi \bar{\mathcal{E}}_\xi}{(\mathcal{E} + \bar{\mathcal{E}})^2} \ , \tag{3.3}$$

where $\xi = z + i\rho$. In terms of the $SL(2, \mathbb{R})/SO(2)$-valued matrix

$$G = \frac{1}{\mathcal{E} + \bar{\mathcal{E}}} \begin{pmatrix} 2 & i(\mathcal{E} - \bar{\mathcal{E}}) \\ i(\mathcal{E} - \bar{\mathcal{E}}) & 2\mathcal{E}\bar{\mathcal{E}} \end{pmatrix} \ . \tag{3.4}$$

the Ernst equation may be equivalently rewritten as follows:

$$(\rho G_\rho G^{-1})_\rho + (\rho G_z G^{-1})_z = 0 \ . \tag{3.5}$$

The relationship between solutions of Schlesinger system and Ernst equation was revealed in [26]:

Theorem 3.1 *Let $\{A_j\}$ be some solution of the Schlesinger system (2.5) and $\Psi(\gamma)$ be related solution of equation (2.1) satisfying the following conditions:*

$$\Psi^t(\frac{1}{\gamma})\Psi(0)^{-1}\Psi(\gamma) = I \ , \tag{3.6}$$

$$\Psi(-\bar{\gamma}) = \overline{\Psi(\gamma)} \ . \tag{3.7}$$

Let in addition $\gamma_j = \gamma(\lambda_j, \xi, \bar\xi)$, $\lambda_j \in \mathbb{C}$ for all j, with function $\gamma(\lambda, \xi, \bar\xi)$ given by

$$\gamma = \frac{2}{\xi - \bar\xi}\left\{\lambda - \frac{\xi + \bar\xi}{2} + \sqrt{(\lambda - \xi)(\lambda - \bar\xi)}\right\} . \tag{3.8}$$

Then function

$$G(\xi, \bar\xi) \equiv \Psi(\gamma = 0, \xi, \bar\xi) \tag{3.9}$$

satisfies Ernst equation (3.5). Metric coefficient e^{2k} of the line element (3.1) coincides with the τ-function of the Schlesinger system up to explicitly computable factor: follows:

$$e^{2k} = C \prod_{j=1}^{N} \left\{\frac{\partial \gamma_j}{\partial \lambda_j}\right\}^{\mathrm{tr} A_j^2/2} \tau , \tag{3.10}$$

where C is an arbitrary constant.

Exploiting the relationship between Schlesinger system and Ernst equation given by the theorem 3.1, and making use of theta-functional solutions of the Schlesinger system provided by the theorem 2.1, we can obtain solutions of Ernst equation in terms of theta-functions. The necessary additional work to do is to choose the parameters of the construction (i.e. the constants λ_j and vectors $\mathbf{p}$, $\mathbf{q}$) to provide the constraints (3.6) and (3.7). To get these constraints fulfilled we have to assume that the curve $\mathcal{L}$ is invariant under the holomorphic involution σ acting on every sheet of $\mathcal{L}$ as $\gamma \to \gamma^{-1}$, and anti-holomorphic involution μ acting on every sheet of $\mathcal{L}$ as $\gamma \to -\bar\gamma$. Constraints (3.6) and (3.7) turn out to be compatible with each other only if the genus g is odd:

$$g = 2g_0 - 1 . \tag{3.11}$$

Let us enumerate the branch points γ_j, $j = 1, \dots, 4g_0$ in such an order that $\gamma_j = \gamma_{j+2g_0}^{-1}$, $j = 1, \dots, 2g_0$ and for some $k \leq g_0$ we have

$$\gamma_j \in i\mathbb{R} , \quad 1 \leq j \leq 2k ; \qquad \gamma_{2j+1} + \bar\gamma_{2j+2} = 0 , \quad 2k+1 \leq j \leq 2g_0 - 1.$$

Choosing the canonical basis of cycles on $\mathcal{L}$ in appropriate way [33], we can easily derive the conditions on constants $\mathbf{p}$ and $\mathbf{q}$ imposed by reductions (3.6) and (3.7) (see [33]). Then relation (3.9) allowes then to express the Ernst potential in terms of theta-functions associated to curve $\mathcal{L}$. However, much simpler description may be acheived if we make use of invariance of curve $\mathcal{L}$ under involution σ and exploit the spectral parameter λ from (3.8) instead of γ. Curve $\mathcal{L}$ may be realised as fourfold covering of λ-plane. Four sheets of this covering will be enumerated in such a way that the involution σ interchanges the sheets $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$; involution $*$ interchanges the sheets $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$.

Now we consider the new hyperelliptic curve $\mathcal{L}_0$ of genus g_0 defined by equation

$$\nu^2 = (\lambda - \xi)(\lambda - \bar\xi) \prod_{j=1}^{2g_0} (\lambda - \lambda_j) . \tag{3.12}$$

We draw the "moving" branch cut between branch points ξ and $\bar\xi$ and g_0 immovable branch cuts $[\lambda_{2j-1}, \lambda_{2j}]$. The canonical basis of cycles on $\mathcal{L}_0$ is chosen such that cycle

a_j encircles the branch cut $[\lambda_{2j-1}, \lambda_{2j}]$. Cycle b_j starts on the left bank of the cut $[\xi, \bar{\xi}]$, goes on the second sheet via the branch cut $[\lambda_{2j-1}, \lambda_{2j}]$ and returnes back.

Introduce the dual basis $dV = (dV_1, \ldots, dV_{g_0})^t$ of holomorphic 1-forms on $\mathcal{L}_0$ by

$$dV_j = \frac{1}{\nu}\sum_{k=1}^{g_0}(\mathcal{A}_0^{-1})_{jk}\lambda^{k-1}d\lambda \;, \qquad j = 1, \ldots, g_0 \;, \tag{3.13}$$

where

$$(\mathcal{A}_0)_{kj} \equiv \oint_{a_j^0} \frac{\lambda^{k-1}d\lambda}{\nu} \;, \qquad j,k = 1, \ldots, g_0 \tag{3.14}$$

The matrix of b-periods of $\mathcal{L}_0$ will be denoted by $\mathbf{B}_0$.

Curve $\mathcal{L}$ is twofold non-ramified covering Π of $\mathcal{L}_0$, such that points of $\mathcal{L}$ related by the involution σ project onto the same point of $\mathcal{L}_0$. Anti-involution μ inherited from $\mathcal{L}$ acts on every sheet of $\mathcal{L}_0$ as $\lambda \to \bar{\lambda}$.

Reduction (3.6) allows to alternatively express function Ψ in terms of theta-functions associated to the curve $\mathcal{L}_0$. Namely, denote by Ω_λ the neighbourhood of the point $\lambda = \infty$ being the projection of the domain Ω_γ into λ-plane. Define function $\Phi_0(\lambda \in \Omega_\lambda)$ in the subdomain Ω_λ of the first sheet of $\mathcal{L}_0$ by the following formula:

$$\Phi_0(\lambda \in \Omega_\lambda) = \begin{pmatrix} \varphi_0(\lambda) & \varphi_0(\lambda^*) \\ \psi_0(\lambda) & \psi_0(\lambda^*) \end{pmatrix}, \tag{3.15}$$

where involution $*$ inherited on $\mathcal{L}_0$ from $\mathcal{L}$ interchanges the λ-sheets of $\mathcal{L}_0$;

$$\varphi_0(\lambda) = \Theta\begin{bmatrix}\mathbf{r}\\ \mathbf{s}\end{bmatrix}\left(\int_\xi^\lambda dV \Big| \mathbf{B}_0\right) \;, \qquad \psi_0(\lambda) = \Theta\begin{bmatrix}\mathbf{r}\\ \mathbf{s}\end{bmatrix}\left(\int_{\bar{\xi}}^\lambda dV \Big| \mathbf{B}_0\right) \;, \tag{3.16}$$

$$\varphi_0(\lambda^*) = -i\Theta\begin{bmatrix}\mathbf{r}\\ \mathbf{s}\end{bmatrix}\left(-\int_\xi^\lambda dV \Big| \mathbf{B}_0\right) \;, \qquad \psi_0(\lambda^*) = i\Theta\begin{bmatrix}\mathbf{r}\\ \mathbf{s}\end{bmatrix}\left(-\int_{\bar{\xi}}^\lambda dV \Big| \mathbf{B}_0\right) \;, \tag{3.17}$$

for $\lambda \in \Omega_\lambda$. Constant vectors $\mathbf{r},\ \mathbf{s} \in \mathbb{C}^{g_0}$ satisfy the following reality conditions:

$$\mathbf{r} \in \mathbb{R}^{g_0} \;, \tag{3.18}$$

$$\Re \mathbf{s}_j = \sum_{l=1}^{g_0} \frac{\mathbf{r}_l}{2} \;, \qquad 1 \le j \le k \;; \qquad \Re \mathbf{s}_j = \sum_{l=1,\, l\neq j}^{g_0} \frac{\mathbf{r}_l}{2} \;, \qquad k+1 \le j \le g_0 \;. \tag{3.19}$$

which may be equivalently rewritten as

$$\Re(\mathbf{B}_0\mathbf{r} + \mathbf{s}) = 0 \;. \tag{3.20}$$

It turns out that if we choose basic cycles on $\mathcal{L}_0$ and $\mathcal{L}$ in consistent way, namely, such that [33]:

$$\Pi(a_1) = -(a_1^0 + \ldots + a_{g_0}^0) \;, \qquad \Pi(b_1) = -2b_1^0 \;, \qquad \Pi(a_j) = a_j^0 \;, \qquad \Pi(b_j) = b_j^0 - b_1^0 \;,$$

and, moreover, impose the following relations between constants $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}, \mathbf{s}$:

$$p_1 = -\sum_{l=1}^{g_0} r_l \;, \quad q_1 = -2s_1 \;,$$

$$p_j = -p_{j+g_0-1} = r_j \ , \quad q_j = -p_{j+g_0-1} = s_j - s_1 \ , \quad 2 \leq j \leq g_0 \ , \tag{3.21}$$

the function Ψ (2.14) in Ω_λ may alternatively be written as follows:

$$\Psi(\lambda \in \Omega_\lambda) = \sqrt{\frac{\det \Phi_0(\infty^1)}{\det \Phi_0(\lambda)}} \Phi_0^{-1}(\infty^1)\Phi_0(\lambda) \ . \tag{3.22}$$

Again, we extend it to the universal covering X by analytical continuation to get function $\Psi(P \in X)$. The proof of coincidence of funtions (3.22) and (2.14) may be obtained by verifying the coincidence of monodromies of these functions around the singularities [33].

Now we are in position to formulate the following statement:

Theorem 3.2 *Let $\mathbf{r}, \mathbf{s} \in \mathbb{C}^{g_0}$ be arbitrary constant vectors satisfying reality conditions (3.18), (3.19). Then the function*

$$\mathcal{E}(\xi, \bar{\xi}) = \frac{\Theta \left[\begin{smallmatrix}\mathbf{r}\\ \mathbf{s}\end{smallmatrix}\right] \left(\int_\xi^{\infty^1} dV \Big| \mathbf{B}_0 \right)}{\Theta \left[\begin{smallmatrix}\mathbf{r}\\ \mathbf{s}\end{smallmatrix}\right] \left(- \int_\xi^{\infty^1} dV | \mathbf{B}_0 \right)} \tag{3.23}$$

solves the Ernst equation (3.2).

Proof. One can represent matrix $\Psi(\lambda = \infty^1)$, given by (3.22), in the form (3.4) with

$$\mathcal{E} = -i \frac{\varphi_0(\infty^1)}{\varphi_0(\infty^2)} \ ,$$

which leads to (3.23) after substitution of (3.16).

Remark 3.1 [21, 33] It turns out that, up to trivial limiting prcedure, class of solutions described by simple expression (3.23) coincides with the class of algebro-geometric solutions of Ernst equation originally obtained in [17] in more complicated form.

Now it arises the non-trivial problem of integration of equations (3.3) for the metric coefficients F and k. It turns out that the arising quadratures may be explicitly resolved; for metric coefficient $F(\xi, \bar{\xi})$ we get the following formula [33]:

$$F = \frac{2}{\Re \mathcal{E}} \Im \left\{ \sum_{j=1}^{g_0} (\mathcal{A}_0^{-1})_{g_0 j} \frac{\partial}{\partial z_j} \ln \Theta \left[\begin{smallmatrix}\mathbf{r}\\ \mathbf{s}\end{smallmatrix}\right] \left(- \int_\xi^{\infty^1} dV \Big| \mathbf{B}_0 \right) \right\} \tag{3.24}$$

where matrix $\mathcal{A}_0$ is the matrix of a-periods of holomorphic 1-forms (3.14); $\frac{\partial \Theta}{\partial z_j}$ denotes derivative of theta-function with respect to its jth argument.

More complicated problem of determination of coefficient $k(\xi, \bar{\xi})$ can be resolved using the expression (2.17) for the τ-function of Schlesinger system and relation (3.10) between the τ-function and the coefficient e^{2k} (remind that for algebro-geometric solutions of the Schlesinger system described in the previous section $\mathrm{tr} A_j^2 = 1/8$):

$$e^{2k} = \frac{\Theta \left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right] (0|\mathbf{B})}{\sqrt{\det \mathcal{A}_0}} \prod_{j=1}^{2g_0} |\lambda_j - \xi|^{-1/4} \ , \tag{3.25}$$

where $\mathbf{B}$ is the matrix of b-periods of curve $\mathcal{L}$ (2.7); vectors $\mathbf{p}, \mathbf{q} \in \mathbb{C}^{2g_0-1}$ are expressed in terms of vectors $\mathbf{r}, \mathbf{s} \in \mathbb{C}^{g_0}$ via relations (3.21).

To get (3.25) we should also make use of the following simple relation between determinants of matrices of a-periods of holomorphic 1-forms of curves $\mathcal{L}$ and $\mathcal{L}_0$:

$$\det \mathcal{A} = const\, \rho^{g_0^2} \det \mathcal{A}_0 \,, \tag{3.26}$$

where the constant is independent of $(\xi, \bar{\xi})$.

As usual, degeneration $\lambda_{2j+1},\ \lambda_{2j+1} \to \kappa_j \in \mathbb{R}$ of curve $\mathcal{L}_0$ into genus zero curve with double points leads after appropriate choice of constants $\mathbf{r}, \mathbf{s}$ to the family of Belinskii-Zakharov multisoliton solutions. In particular, to get Kerr-NUT soultion one has to take $g_0 = 2$ and

$$r_j = \frac{n_j}{2}\,; \qquad s_j = \frac{n_j}{4} + i\alpha_j, \qquad \alpha_j \in \mathbb{R}\,, \qquad n_j = \pm 1\,, \qquad j = 1, 2\,.$$

Then (3.23) turns into [17]:

$$\mathcal{E} = \frac{1-\Gamma}{1+\Gamma}\,, \qquad \Gamma^{-1} = \frac{i(a_1 - a_2)}{1 - a_1 a_2 + i(a_1 + a_2)} X + \frac{1 + a_1 a_2}{1 - a_1 a_2 + i(a_1 + a_2)} Y\,,$$

where $a_j = n_j e^{-2\pi i \alpha_j}$, $j = 1, 2$, and

$$X = \frac{1}{\kappa_1 - \kappa_2}\Big\{\sqrt{(\kappa_1 - \xi)(\kappa_1 - \bar{\xi})} + \sqrt{(\kappa_2 - \xi)(\kappa_2 - \bar{\xi})}\Big\}$$

$$Y = \frac{1}{\kappa_1 - \kappa_2}\Big\{\sqrt{(\kappa_1 - \xi)(\kappa_1 - \bar{\xi})} - \sqrt{(\kappa_2 - \xi)(\kappa_2 - \bar{\xi})}\Big\}$$

are prolate ellipsoidal coordinates.

4 Self-dual $SU(2)$ invariant Einstein metrics and an example of Frobenius manifold

Here we are going to describe two other applications of the simplest non-trivial (elliptic) version of construction of section 2. Take $N = 4$; as is well-known, the Schlesinger system is in this case equivalent to Painlevé 6 equation. In sect. 2 we got a class of solutions of the four-pole Schlesinger system satisfying conditions $\mathrm{tr} A_j^2 = 1/8$ and $\sum_{j=1}^4 A_j = 0$. Without loss of generality we can put $\gamma_4 = \infty$. Then the four-pole Schlesinger system may be reformulated in terms of new dependent variables $\Omega_1, \Omega_2, \Omega_3$ defined by

$$\Omega_1^2 = -(\frac{1}{8} + \mathrm{tr} A_1 A_2)\,, \qquad \Omega_2^2 = \frac{1}{8} + \mathrm{tr} A_2 A_3\,, \qquad \Omega_3^2 = \frac{1}{8} + \mathrm{tr} A_1 A_3 \tag{4.1}$$

and new independent variable - the module

$$x = \frac{\gamma_3 - \gamma_1}{\gamma_3 - \gamma_2} \tag{4.2}$$

of curve $\mathcal{L}$. If we choose signes of functions Ω_j in appropriate way, they satisfy the following system of equations [13]:

$$\frac{d\Omega_1}{dx} = -\frac{\Omega_2 \Omega_3}{x(1-x)}\,, \qquad \frac{d\Omega_2}{dx} = -\frac{\Omega_3 \Omega_1}{x}\,, \qquad \frac{d\Omega_3}{dx} = -\frac{\Omega_1 \Omega_2}{1-x}\,, \tag{4.3}$$

together with condition

$$-\Omega_1^2+\Omega_2^2+\Omega_3^2=\frac{1}{4}\ . \tag{4.4}$$

We can mention at least two different geometrical applications of the system (4.3):

1. *Self-dual $SU(2)$-invariant Einstein manifolds.* In the complete local classification of self-dual $SU(2)$-invariant Einstein manifolds (see [13]) the most non-trivial class is described by the following metric on the unit ball:

$$g=F\left\{\frac{dx^2}{x(1-x)}+\frac{\sigma_1^2}{\Omega_1^2}+\frac{(1-x)\sigma_2^2}{\Omega_2^2}+\frac{x\sigma_3^2}{\Omega_3^2}\right\}\ . \tag{4.5}$$

   where σ_j are left-invariant 1-forms on $SU(2)$ satisfying relations $d\sigma_j=\sigma_k\sigma_l$ for any even transposition j,k,l of indeces $1,2,3$; functions F and Ω_j depend only on "euclidean time" x.

   Self-duality of related Weyl tensor, together with assumption that this metric satisfies Einstein equations with cosmological constant Λ, implies (see [31]) system (4.3), (4.4) and the following formula for conformal factor F:

$$F=-\frac{1}{4\Lambda}\frac{8x\Omega_1^2\Omega_2^2\Omega_3^2+2\Omega_1\Omega_2\Omega_3(x(\Omega_1^2+\Omega_2^2)-(1-4\Omega_3^2)(\Omega_2^2-(1-x)\Omega_1^2))}{(x\Omega_1\Omega_2+2\Omega_3(\Omega_2^2-(1-x)\Omega_1^2))^2}\ . \tag{4.6}$$

2. *Three-dimensional Frobenius manifolds.* Consider the following metric tensor on three-dimensional manifold M:

$$g_0=-\Omega_1^2d\gamma_1^2+\Omega_2^2d\gamma_2^2+\Omega_3^2d\gamma_3^2 \tag{4.7}$$

   Let us assume that the metric is flat i.e. associated Riemann tensor vanishes. Assume in addition that all variables Ω_j depend on γ_j only via variable x (4.2). Then functions $\Omega_j(x)$ satisfy the system (4.3) (see [32, 30]). According to Dubrovin [32], in this case M carries the structure of Frobenius manifold with spectrum $(-\frac{1}{4},0,\frac{1}{4})$ and metric (4.7) corresponds to some solution of Witten-Dijkgraaf-Verlinde-Verlinde equation. Variables γ_j are called the canonical coordinates on M.

It is worth to notice that all solutions of the system (4.3), (4.4) are known for at least several years: for example, they can be obtained via Okamoto transformations [34] from classical Picard solution of Painlevé 6 equation; independently they were discovered in [13]. However, the construction of sect.2 allows to find the general solution of (4.3), (4.4) in the following pretty simple form [29]:

$$\Omega_1=\frac{e^{-\pi ip}}{2\pi i\vartheta_2\vartheta_4}\frac{\frac{d}{dq}\vartheta\begin{bmatrix}p+1/2\\ q+1/2\end{bmatrix}}{\vartheta\begin{bmatrix}p\\ q\end{bmatrix}}\ ,\qquad \Omega_2=\frac{1}{2\pi\vartheta_2\vartheta_3}\frac{\frac{d}{dq}\vartheta\begin{bmatrix}p+1/2\\ q\end{bmatrix}}{\vartheta\begin{bmatrix}p\\ q\end{bmatrix}}\ ,$$

$$\Omega_3=\frac{e^{-\pi ip}}{2\pi i\vartheta_3\vartheta_4}\frac{\frac{d}{dq}\vartheta\begin{bmatrix}p\\ q+1/2\end{bmatrix}}{\vartheta\begin{bmatrix}p\\ q\end{bmatrix}}\ , \tag{4.8}$$

where

$$\vartheta_2\equiv\vartheta\begin{bmatrix}1/2\\ 0\end{bmatrix}(0|\mathbf{B})\ ,\quad \vartheta_3\equiv\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(0|\mathbf{B})\ ,\quad \vartheta_4\equiv\vartheta\begin{bmatrix}0\\ 1/2\end{bmatrix}(0|\mathbf{B}) \tag{4.9}$$

are standard theta-constants and $\vartheta\left[{p \atop q}\right]$ stands for $\vartheta\left[{p \atop q}\right](0|\mathbf{B})$. Obviously, in the elliptic case period $\mathbf{B}$ depends only on the module x of curve $\mathcal{L}$.

In addition to this general two-parametric family, there exist a one-parametric family of solutions of (4.3), (4.4) which may be obtained from (4.8) in the limit $p, q \to 1/2$.

Substitution of expressions (4.8) into the formula for conformal factor (4.6) gives the following simple result [29]:

$$F = -\frac{2\pi^2\vartheta_3^4}{\Lambda\frac{d}{dq}\log\vartheta\left[{p \atop q}\right]}\Omega_1\Omega_2\Omega_3 \ .$$

In the paper [29] formulas (4.8) were derived as a result of rather complicated calculation starting from expression (2.17) for the tau-function of Schlesinger system. However, they can also be derived in straightforward way from the formulas (2.11), (2.12), (2.13), (2.14) for function $\Psi(\gamma)$, if in the process of calculation of $\mathrm{tr}A_jA_k$ we choose free parameters γ_φ and γ_ψ as follows: $\gamma_\varphi = \gamma_j$, $\gamma_\psi = \gamma_k$.

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